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Schaft, A.J. van der; Willems, J.C.

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STOCHASTIC REALIZATION OF SPECTRAL DENSITY MATRICES WHICH MAY POSSESS SYMMETRIES

A.J. van der Schaft
Twente University of Technology, Dept. of Applied Mathematics
P.O.Box 217, 7500 AE Enschede, The Netherlands

J.C. Willems
University of Groningen, Mathematics Institute
P.O. Box 800, 9700 AV Groningen, The Netherlands

Abstract

A new procedure for stochastic realization of spectral density matrices as proposed in [6] is briefly reviewed. Afterwards we show how this approach may be used such that external symmetries are reflected in the stochastic realization.

1. A new procedure for stochastic realization and spectral factorization of spectral density matrices

In this section we will briefly review the procedure for stochastic realization as proposed in our paper [6]. We study the (weak) *stochastic realization* problem [3]:
Given a stationary zero mean gaussian p-dimensional vector process $y = \{y_t,\ t \in \mathbb{R}\}$ with a $p \times p$ rational spectral density matrix $\Phi(s)$, construct a stationary zero mean Gauss-Markov n-dimensional vector process $x = \{x_t,\ t \in \mathbb{R}\}$ and a $p \times n$ matrix C such that the spectral density of Cx equals $\Phi(s)$. We call (x,C) a stochastic realization of $\Phi(s)$.

The above problem of constructing x and C is equivalent to the problem of generating $\Phi(s)$ by passing white noise through a linear system (the shaping filter). Then we have to construct an $n \times n$ matrix A, an $n \times n$ matrix $Q = Q^T > 0$, and an $p \times n$ matrix C such that the stochastic differential equation

$$dx = Ax\,dt + B\,dw \quad , \quad x(0) = x_o \tag{1}$$

with A asymptotically stable, x_o zero mean gaussian, $E\{x_o x_o^T\} = Q$, $AQ + QA^T = -BB^T$ and $w := \{w_t,\ t \in [0,\infty)\}$ a normalized Wiener process independent of x_o, yields a vector process Cx with spectral density equal to $\Phi(s)$. We also call (A,Q,C) a stochastic realization of $\Phi(s)$. Furthermore we require n, the dimension of the state space of a stochastic realization, to be as small as possible. If n is such we call (x,C), or (A,Q,C), a *minimal* stochastic realization of $\Phi(s)$.

It is easy to see the spectral density of Cx with x generated by (1) is equal to $C(Is-A)^{-1}BB^T(-Is-A^T)^{-1}C^T$. Hence the stochastic realization problem for $\Phi(s)$ is also equivalent to finding a transfer matrix W(s), analytic in $\mathrm{Re}\, s > 0$, such that $\Phi(s) = W(s)W^T(-s)$. This is the *spectral factorization* problem.

The usual procedure for stochastic realization starts with the *partial fraction expension* $\Phi(s) = Z(s) + Z^T(-s)$, with Z(s) analytic in $\mathrm{Re}\, s > 0$. Then one continues by applying the Kalman-Yakubovich-Popov lemma to a minimal realization of the (positive real) transfer matrix Z(s). In a certain sense this approach is natural since Z(s) equals the (two-sided) Laplace transform of the autocorrelation matrix of a process y with spectral density $\Phi(s)$. (Hence if the process is *not* given by $\Phi(s)$ but by its autocorrelation matrix this is certainly the easiest way). On the other hand the partial fraction expansion requires the factorization of (high order) polynomials, which is a nonlinear problem of the same level of difficulty as solving an algebraic Riccati equation. Motivated by this we have proposed in [6] another approach to stochastic realization of $\Phi(s)$, which avoids the partial fraction expansion, and seems also theoretically of interest. However, which of the two approaches is algorithmically most advantageous remains a matter of study.

It is well-known that a rational spectral density matrix $\Phi(s)$ of a process with integrable autocorrelation function is characterized by the following properties

(i) $\Phi(s) = \Phi^T(-s)$ (this is usually called *para-hermitian*)
(ii) $\Phi(s)$ has no poles on the imaginary axis
(iii) $\Phi(i\omega) \geq 0,\ \forall \omega \in \mathbb{R}$
(iv) $\lim_{s\to\infty} \Phi(s) = 0$

Our approach is based on viewing $\Phi(s)$ *as a transfer matrix* and the following key observations.

1. By i) $\Phi(s)$ is a *Hamiltonian transfer matrix* [2,4]. It follows from the state space isomorphism theorem that a minimal realization $(\bar{A},\bar{B},\bar{C})$ of $\Phi(s)$ (i.e. $\Phi(s) = \bar{C}(Is - \bar{A})^{-1}\bar{B}$) will satisfy.

$$\bar{A}^T J + J\bar{A} = 0 \tag{2}$$

$$\bar{B}^T J = \bar{C} \tag{3}$$

for some unique nonsingular matrix J satisfying $J = -J^T$. This already implies that the dimension of the state space of such a minimal realization is even, say 2n.

2. By iii) the transfer matrix $\Phi(s)$ is *passive* (for any input-output pair which corresponds to a closed path in state space, the $\mathcal{L}_2$ inner product of input and output is nonnegative [9]). In fact iii) implies the existence of a $\Sigma = \Sigma^T$, $\det \Sigma \neq 0$, such that

$$\bar{A}^T\Sigma + \Sigma\bar{A} \leq 0 \tag{4}$$

$$\bar{B}^T\Sigma = \bar{C} \tag{5}$$

3. From (2) and (3) it follows that if Σ satisfies (4) and (5), then so does $J\Sigma^{-1}J$. Since the set of Σ's satisfying (4) and (5) is convex and compact, and the map $\Sigma \to J\Sigma^{-1}J$ on the space of nonsingular symmetric matrices is continuous, we obtain by an application of Brouwer's fixed point theorem that there also exist Σ satisfying (4) and (5) *and*

$$\Sigma = J\Sigma^{-1}J \tag{6}$$

4. Now consider a solution Σ of (4), (5), (6) and the unique solution J of (2), (3). It can be proved that there exist bases of $\mathbb{R}^{2n}$ such that

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \tag{7}$$

In such a basis it follows from (2), (3), (4), (5) that $\bar{A}$, $\bar{B}$ and $\bar{C}$ have the form

$$\bar{A} = \begin{bmatrix} F & -P \\ -R & -F^T \end{bmatrix}; \quad \bar{B} = \begin{bmatrix} 0 \\ H^T \end{bmatrix}, \quad \bar{C} = (H \quad 0) \tag{8}$$

with $P = P^T > 0$, $R = R^T \geq 0$. Moreover it follows from controllability of $(\bar{A}, \bar{B})$ that (F, P) is controllable.

We now have the basic ingredients for a stochastic realization of $\Phi(s)$. Consider the Riccati equation associated to $\bar{A}$ in (8):

$$F^TK + KF - K^TPK + R = 0 \tag{9}$$

It follows from observation 4, and property ii) of $\Phi(s)$ that (9) has a maximal symmetric solution K^+ and a minimal symmetric solution K^-, such that $K^+ - K^-$ is positive definite. By application of the symplectic transformation $\bar{K} = \begin{pmatrix} I & 0 \\ -K^+ & I \end{pmatrix}$ to $\bar{A}$, $\bar{B}$, $\bar{C}$ we obtain

$$\bar{K}\bar{A}\bar{K}^{-1} = \begin{bmatrix} F-PK^+ & -P \\ 0 & -(F-PK^+)^T \end{bmatrix}, \quad \bar{K}\bar{B} = \begin{bmatrix} 0 \\ H^T \end{bmatrix}; \bar{C}\bar{K}^{-1} = (H \; 0) \tag{10}$$

Now define $A := F-PK^+$, $C := H$, $Q = (K^+ - K^-)^{-1}$ and write $P = BB^T$. Then it is easy to see that

$$\bar{C}(Is - \bar{A})^{-1}\bar{B} = C(Is - A)^{-1}BB^T(-Is - A^T)^{-1}C^T \tag{11}$$

and that the stochastic system

$$\begin{aligned} dx &= Ax\,dt + B\,dw, \quad x(0) = x_0 \\ \tilde{y} &= Cx \end{aligned} \tag{12}$$

with $E\{x_0x_0^T\} = Q$, generates a process $\tilde{y}$ with spectral density $\Phi(s)$.

With respect to the computational side of the above procedure we make the following remarks (see [6] for details). Having fixed a Σ satisfying (4), (5), there exists an algorithm to compute a Σ satisfying (4), (5) *and* (6). The basis referred to in observation 3 can be directly computed from J and Σ. In fact $\mathrm{Ker}(I-J^{-1}\Sigma)$ is the upper half of the state space, while $\mathrm{Ker}(I + J^{-1}\Sigma)$ is the lower half. What remains to be done is the solution of the Riccati equation (9), especially K^+. We notice that after having fixed a Σ satisfying (4), (5), (6) the stochastic realization (12) is uniquely determined up to a state space transformation. The possible freedom in choosing a Σ satisfying (4), (5), (6) corresponds to the partioning of the transmission zeros of $\Phi(s)$ into the zeros of $W(s)$ and $W^T(-s)$ or, equivalently, to the freedom in the choice of the covariance Q of the shaping filter.

2. Symmetries

Let y be again a zero mean gaussian vector process with spectral density $\Phi(s)$. In this section we will show how we can adapt the algorithm for stochastic realization of $\Phi(s)$ given in the previous section in such a way that symmetry properties of the stochastic process y are reflected in its state space realization (x,C). We call two p-dimensional vector processes y and y' (stochastically) equivalent, denoted by $y \sim y'$, if the probability distributions of the vectors $(y_{t_1}, y_{t_2}, \ldots, y_{t_k})$ and $(y'_{t_1}, y'_{t_2}, y'_{t_k})$ are equal for all choices of $t_1, t_2, \ldots, t_k$. Equivalently, $y \sim y'$ if y and y' haven the same spectral densities or autocorrelation functions.

Definition 1
Let $y := \{y_t, t \in \mathbb{R}\}$ be a p-dimensional vector process. An *(external) symmetry* for y is a nonsingular linear map $U : \mathbb{R}^p \to \mathbb{R}^p$ such such that $Uy \sim y$. Let (x, C) be an n-dimensional realization of y. A symmetry for the combined process (x,y) is a pair (S,U), with $U : \mathbb{R}^p \to \mathbb{R}^p$ and $S : \mathbb{R}^n \to \mathbb{R}^n$ nonsingular linear maps, such that $(Sx, Uy) \sim (x,y)$. We will call such an $S : \mathbb{R}^n \to \mathbb{R}^n$ (hence $Sx \sim x$) an *(internal) symmetry*.

First we will treat the case that y is given by an autocorrelation function $R(t) = E\{y(t)y^T(0)\}$.

Proposition 2
Let y be a p-dimensional vector process with autocorrelation function R. Let U be an (external) symmetry for y. Then there exists a minimal realization (x,C) of R and, given this realization, a uniquely determined S such that (S,U) is a symmetry for (x,y). This implies that we can choose the covariance Q of a minimal realization (A,Q,C) such that

$$SAS^{-1} = A, \quad SQS^T = Q, \quad CS^{-1} = UC. \tag{13}$$

Proof: From $y \sim Uy$ it follows that

$$R(t) = E\{y(t)y^T(0)\} = E\{Uy(t)y^T(0)U^T\} = UR(t)U^T.$$

Let (F,G,H) be a minimal n-dimensional realization of R, i.e. $R(t) = He^{Ft}G$. Then by the state space isomorphism theorem, $R(t) = UR(t)U^T$ implies the existence of a nonsingular $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$SFS^{-1} = F, \; SG = GU^T, \; HS^{-1} = UH \tag{14}$$

The set of solution $Q > 0$ of

$$FQ + QF^T \leq 0, \; QH^T = G \tag{15}$$

is convex and compact. Moreover it follows from (14) that if Q is a solution of (15), so is SQS^T. Therefore since the map $Q \to SQS^T$ (viewed as a map on the symmetric matrices) is continuous, there exists a $Q > 0$ satisfying (15) such that $Q = SQS^T$. Take $A = F$, $C = H$, and we have obtained a minimal realization (A,Q,C) satisfying (13).

□

Therefore we see that the general statement that an external symmetry for y implies the existence of an internal symmetry for a certain class of minimal realizations of y, is easily proved if we use the autocorrelation function of y. We will now translate the property of symmetry to spectral density matrices and show how a minimal stochastic realization possessing a symmetry can be obtained using the algorithm proposed in Section 1.

Theorem 3
Let y be a p-dimensional vector process given by a spectral density matrix $\Phi(s)$. U is a symmetry for y iff $U\Phi(s)U^T = \Phi(s)$. Then there exists a minimal stochastic realization (x,C) of $\Phi(s)$ and a (unique) S such that (S,U) is a symmetry for (x,y). Equivalently, if this realization is given by (A,Q,C), equations (13) hold.

Proof. It can be easily seen that

$$\{y \sim Uy\} \Leftrightarrow \{R(t) = U\,R(t)U^T\} \Leftrightarrow \{\Phi(s) = U\,\Phi(s)U^T\}.$$

Let now $(\bar{A},\bar{B},\bar{C})$ be a minimal 2n-diensional realization of $\Phi(s)$, satisfying (see (2), (3))

$$\bar{A}^TJ + J\bar{A} = 0, \; \bar{B}^TJ = \bar{C}, \text{ with } J = -J^T \tag{16}$$

Since $\Phi(s) = U\Phi(s)U^T$, the state space isomorphism theorem implies the existence of a unique nonsingular $\bar{S} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that

$$\bar{S}\bar{A}\bar{S}^{-1} = \bar{A}, \; \bar{S}\bar{B} = \bar{B}U^T, \; C\bar{S}^{-1} = U\bar{C} \tag{17}$$

Moreover, $\bar{S}$ is necessarily sympletic, i.e. $\bar{S}^TJ\bar{S} = J$. Indeed, it follows from (16) and (17) that for all $r \geq 0$ $J\bar{S}\bar{A}^r\bar{B} = J\bar{A}^r\bar{S}\bar{B} = (-1)^r(\bar{A}^T)^rJ\bar{B}U^T =$
$= -(-1)^r(\bar{A}^T)^r\bar{C}^TU^T = -(-1)^r(\bar{A}^T)^r(\bar{S}^T)^{-1}\bar{C}^T =$
$= -(-1)^r(\bar{S}^T)^{-1}(\bar{A}^T)^rJJ^{-1}\bar{C}^T = (\bar{S}^T)^{-1}J\bar{A}^r\bar{B}$

and therefore, by controllability of $(\bar{A},\bar{B})$, we see that $J\bar{S} = (\bar{S}^T)^{-1}J$, or $\bar{S}^T J\bar{S} = J$. It can now be checked (using (17)) that if $\Sigma = \Sigma^T$ is a solution of

$$\bar{A}^T\Sigma + \Sigma\bar{A} < 0, \quad \bar{B}^T\Sigma = \bar{C} \tag{18}$$

then so is $\bar{S}^T \Sigma \bar{S}$. From the passivity of $\Phi(s)$ it follows that the set of Σ's, satisfying $\Sigma = \Sigma^T$, $\det \Sigma \neq 0$ and (18), is nonempty. Moreover this set is convex and compact. Therefore, since the map $\Sigma \to \bar{S}^T \Sigma \bar{S}$ (with $\Sigma = \Sigma^T, \det \Sigma \neq 0$) is continuous there exists such a Σ satisfying (18) and $\Sigma = \bar{S}^T \Sigma \bar{S}$. However the set of Σ's satisfying (18) and $\Sigma = \bar{S}^T\Sigma\bar{S}$ is again convex and compact. Furthermore if Σ is a solution of (18) and $\Sigma = \bar{S}^T \Sigma \bar{S}$, so is $J\Sigma^{-1}J$ (this follows from (16) and $\bar{S}^T J\bar{S} = J$). Hence, since the map $\Sigma \to J\Sigma^{-1}J$, with $\Sigma = \Sigma^T$ and $\det \Sigma \neq 0$, is continuous, there exists a $\Sigma = \Sigma^T$, $\det \Sigma \neq 0$, satisfying (18) as well as $\Sigma = \bar{S}^T\Sigma\bar{S}$ and $\Sigma = J\Sigma^{-1}J$.

Given such a Σ, there exists a basis (see equation (7)) of $\mathbb{R}^{2n}$ such that

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}.$$

From $\bar{S}^T\Sigma\bar{S} = \Sigma$ and $\bar{S}^T J\bar{S} = J$ it follows that $\bar{S}J^{-1}\Sigma = J^{-1}\Sigma\bar{S}$. Therefore $\bar{S}$ commutes with $J^{-1}\Sigma$ and hence leaves the positive and negative eigenspaces of $J^{-1}\Sigma$ (corresponding respectively to the n eigenvalues of $J^{-1}\Sigma$ which are +1 and the n eigenvalues which are -1) invariant. Since $\bar{S}$ is symplectic, we obtain that in this basis $\bar{S}$ has the form

$$\bar{S} = \begin{bmatrix} S & 0 \\ 0 & (S^T)^{-1} \end{bmatrix}, \text{ with } S \text{ nonsingular.} \tag{19}$$

Now $\bar{A}$,$\bar{B}$ and $\bar{C}$ can be written in this basis as in equation (8). Then from (17) and (19) it follows that

$$SFS^{-1} = F,\ SPS^T = P,\ S^TRS = R,\ HS^{-1} = UH \tag{20}$$

We now look at the Riccati equation associated to $\bar{A}$

$$F^TK + KF - K^TPK + R = 0 \tag{21}$$

We state separately

Lemma 4
Let K^+ and K^- be the maximal and minimal solution of (21), and let S satisfy (20). Then $S^TK^+S = K^+$ and $S^TK^-S = K^-$.

Proof
Consider the usual partial ordering on symmetric matrices: $K_1 \geq K_2 \Leftrightarrow K_1 - K_2$ is nonnegative definite. Under this orderning the set $\underline{K}$ of symmetric solutions of (18) is partially ordered with unique maximum K^+ and unique minimum K^- ([8]). From (20) it follows that if $K \in \underline{K}$ then also $S^TKS \in \underline{K}$. Furthermore the map $K \to S^TKS$ preserves the ordening of $\underline{K}$. Hence the maximum K^+ and the minimum K^- are invariant under this map. ∎

Proof of Theorem 3 continued:
Let us again denote $A:= F - PK^+$, $C:= H$. Then it follows from Lemma 3 and (20) that $SAS^{-1} = A$ and $CS^{-1} = UC$. Moreover it follows from Lemma 3 that $S^T(K^+-K^-)S = K^+-K^-$, and hence, since $Q = (K^+ - K^-)^{-1}$, that $SQS^T = Q$. ∎

Remark
An external symmetry U for a spectral density $\Phi(s)$ can be also interpreted as a *Hamiltonian symmetry* (as defined in [5]) for the internal Hamiltonian transfer matrix $\Phi(s)$. Notice that $\bar{S}$ is an internal Hamiltonian symmetry for $(\bar{A},\bar{B},\bar{C})$, and that S is a symmetry for the Riccati equation (21) (see also [7]).

It immediately follows from the proof of Theorem 3 how we can adapt the algorithm of Section 1 to obtain a minimal stochastic realization (A,Q,C) exhibiting an internal symmetry S if $\phi(s)$ has an external symmetry U. In fact there exists a unique $\bar{S}$ satisfying (17). Then we have to find a Σ satisfying (4), (5), (6) *and* $\Sigma = \bar{S}^T\Sigma\bar{S}$. Taking a basis as in (7) we obtain $\bar{S} = \begin{pmatrix} S & 0 \\ 0 & (S^T)^{-1}\end{pmatrix}$, with S the resulting internal symmetry. From a computational point of view the easiest way is first to compute a solution Σ of (4), (5) and $\Sigma = \bar{S}^T\Sigma\bar{S}$, and then to take this solution as starting point for the algorithm defined in [6, equation (16)] to obtain a Σ also satisfying (6).

Instead of dealing with a *single* external symmetry it is of interest to consider a *group* of external symmetries U. It then follows from the state space isomorphism theorem that there exists an isomorphic group of symmetries $\bar{S}$ for $(\bar{A},\bar{B},\bar{C})$. In order to obtain a group of internal symmetries S for a stochastic realization (x,C) of $\phi(s)$ it is necessary and sufficient that there exists a Σ satisfying (4), (5) and $\bar{S}^T\Sigma\bar{S}$ for *all* symmetries $\bar{S}$. This is a very interesting problem. If the group G of $\bar{S}$ is abelian, it is clear that such a Σ exists. For take an arbitrary $\bar{S} \in G$. Then the set $\Sigma_{\bar{S}}$ of nonsingular Σ satisfying (4), (5) and $\bar{S}^T\Sigma\bar{S} = \Sigma$ is nonempty, and convex and compact. Take another symmetry $S_1 \in G$, and consider the map $\Sigma \to \bar{S}_1{}^T \Sigma \bar{S}_1$ for $\Sigma \in \Sigma_{\bar{S}}$. Since G is abelian $\bar{S}^T(\bar{S}_1^T \Sigma \bar{S}_1)\bar{S} = \bar{S}^T(\bar{S}^T \Sigma \bar{S})\bar{S}_1 = \bar{S}_1{}^T \Sigma \bar{S}_1$ and it easily follows that $\bar{S}_1{}^T \Sigma \bar{S}_1$ is again in $\Sigma_{\bar{S}}$. Hence by application of Brouwer's fixed point theorem to $\Sigma_{\bar{S}}$ and this map, we conclude that the set of Σ satisfying (4), (5) and $\bar{S}^T \Sigma \bar{S} = \Sigma = \bar{S}_1^T \Sigma \bar{S}_1$ is nonempty.
By continuing this process for a set of generators of G we finally obtain a Σ satisfying (4), (5) and $\bar{S}^T \Sigma \bar{S} = \bar{S}$ for *all* $\bar{S} \in \Sigma$. However this proof does not work for non-abelian groups, and therefore it remains an open problem if for a non-abelian group of symmetries for $\Phi(s)$ there exists a minimal stochastic realization of $\Phi(s)$ displaying an isomorphic group of internal symmetries.

Now we turn attention to a more special kind of symmetry, namely *time-reversibility*.

Definition 5
Let $y: = \{y_t,\ t \in \mathbb{R}\}$ be a p-dimensional vector process. We call y *time-reversible* if $y \sim Ry$ (with R defined by $(Ry)(t):= y(-t)$).

If the stochastic process y is given by its autocorrelation function R, it can be easily seen that time-reversibility is equivalent to $R = R^T$. The following result is then obtained in [10] (see also [1]).

Theorem 6
Let R *be an autocorrelation function satisfying* $R = R^T$. *Then* R *has a minimal stochastic realization* (A,Q,C) *satisfying*

$$A = MA^T,\ CM = C,\ Q = I \tag{22}$$

where M *denotes a signature matrix (i.e. a diagonal matrix with elements* +1 *or* −1 *on the diagonal).*

The above theorem hence effectively says that there exists a realization (x,C) of y such that $(x,y) \sim R(Mx,y)$.

If y is given by its spectral density $\Phi(s)$, then time-reversibility corresponds to $\Phi(s) = \Phi(-s)$ (or, since $\Phi(s) = \Phi^T(-s)$, to $\Phi(s) = \Phi^T(s)$). In the following theorem we give a constructive proof how by using the approach sketched in Section 1 we can also obtain a stochastic realization satisfying (22).

Theorem 7
Let $\Phi(s)$ *be a spectral density satisfying* $\Phi(s) = \Phi(-s)$. *Then there exists a minimal stochastic realization*

(A,Q,C) *of* $\Phi(s)$ *satisfying* $A = MA^TM$, $CM = C$, $Q = I$, *with* M *a signature matrix.*

Proof

Let $(\bar{A},\bar{B},\bar{C})$, be a minimal 2n-dimensional realization of $\Phi(s)$, satisfying $\bar{A}^TJ + J\bar{A} = 0$, $\bar{B}^TJ = \bar{C}$ (see (2.3)). Since $\Phi(s) = \Phi(-s)$, also $(-\bar{A}, -\bar{B}, \bar{C})$ is a minimal realization. Hence by the state space isomorphism theorem, there exists a unique $\bar{S} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that

$$\bar{S}\bar{A}\bar{S}^{-1} = -\bar{A},\ \bar{S}\bar{B} = -\bar{B},\ \bar{C}\bar{S}^{-1} = \bar{C}. \tag{23}$$

Furthermore, in [4] it is proved this $\bar{S}$ also satisfies

$$\bar{S}^T\bar{J}\bar{S} = -J,\ \bar{S}^2 = I. \tag{24}$$

Now look at the solution $\Sigma = \Sigma^T$ of

$$\bar{A}^T\Sigma + \Sigma\bar{A} \preccurlyeq 0,\quad \bar{B}^T\Sigma = \bar{C} \tag{25}$$

If Σ is a solution of (25), so is $-\bar{S}^T\Sigma\bar{S}$ by (23). Then since the map $\Sigma \to -\bar{S}^T\Sigma\bar{S}$ is continuous, and the solution set of (25) is convex and compact there exists a $\Sigma = \Sigma^T$ satisfying (25), such that $\Sigma = -\bar{S}^T\Sigma\bar{S}$. The set of Σ's satisfying (25) *and* $\Sigma = -\bar{S}^T\Sigma\bar{S}$ is again convex and compact. Moreover if Σ satifies (25) and $\Sigma = -\bar{S}^T\Sigma\bar{S}$, so does $J\Sigma^{-1}J$, by $\bar{S}^TJ\bar{S} = -J$ and (2), (3). Since the map $\Sigma \to J\Sigma^{-1}J$ (viewed as a map on the space of symmetric nonsingular matrices), is continuous, this implies the existence of a $\Sigma = \Sigma^T$, $\det\Sigma \neq 0$, satisfying (25) and

$$\Sigma = -S^T\Sigma S\ ,\ \Sigma = J\Sigma^{-1}J \tag{26}$$

From $\Sigma = -\bar{S}^T\Sigma S$ and $J = -\bar{S}^TJ\bar{S}$ it follows that $\bar{S}J^{-1}\Sigma = J^{-1}\Sigma\bar{S}$. Therefore $\bar{S}$ leaves the positive and negative eigenspaces of $J^{-1}\Sigma$ invariant. Hence if we take a basis as in (7), $\bar{S} = \begin{pmatrix} S & 0 \\ 0 & -S^T\end{pmatrix}$, with $S^2 = I$. In this basis $\bar{A}$, $\bar{B}$ and $\bar{C}$ are as in (8). Then (23) implies

$$SFS = -F,\ SPS^T = P,\ S^TRS = R,\ \ HS = H \tag{27}$$

For the Riccati equation (9) associated to $\bar{A}$ we obtain

Lemma 7

Let K^+ and K^- be the maximal and minimal solution of (9), and let S satisfy (27). Then $S^TK^+S = -K^-$ and $S^TK^-S = -K^+$.

Proof

If K satisfies (9), then so does $-S^TKS$, by (27). Furthermore the map $K \to -S^TKS$ reverses the partial ordening on the solution set of (9). Therefore the maximum K^+ is mapped onto the minimum and vice versa.

□

Proof of Theorem 6 continued

Define $F^+: = F - PK^+$, $F^-: = F - PK^-$. It follows from Lemma 7 and (27) that

$$SF^+ = S(F-PK^+) = -FS + PK^- = -F^-S \tag{28}$$

and, since K^+ and K^- are solutions of (9), that:

$$(F^+)^T(K^+ - K^-) + (K^+ - K^-)F^- = 0 \tag{29}$$

(see [8]). Since $K^+ - K^- > 0$ we can find a nonsingular N such that $N^T(K^+ - K^-)N = I$. Then apply the basis-transformation $\bar{N} = \begin{pmatrix} N & 0 \\ 0 & (N^T)^{-1}\end{pmatrix}$. The Riccati equation associated to $\bar{N}\bar{A}\bar{N}^{-1}$ has maximal solution N^TK^+N and minimal solution N^TK^-N. Therefore in this new basis $Q^{-1} = N^T(K^+ - K^-)N = I$. From Lemma 7 it follows that $S^T(K^+ - K^-)S = K^+ - K^-$, and hence in this basis $S^TS = I$, or, since $S^2 = I$, $S = S^T$. Therefore there exists an orthogonal O such that $OSO^T = M$, with M a signature matrix. Apply the basis transformation $\bar{O} = \begin{pmatrix} O & 0 \\ 0 & (O^T)^{-1}\end{pmatrix}$. This transforms $\bar{S} = \begin{pmatrix} S & 0 \\ 0 & -S^T\end{pmatrix}$ into $\begin{pmatrix} M & 0 \\ 0 & -M^T\end{pmatrix}$. Since O is orthogonal it leaves $K^+ - K^-$ invariant and therefore Q remains the identity. Finally write in this new basis $\bar{A}$, $\bar{B}$, $\bar{C}$ as in (8).

Then define $A: = F - PK^+$, $C: = H$, $Q: = I$. Since $K^+ - K^- = I$ (29) yields $(F^+)^T + F^- = 0$, and therefore (28) is equivalent to (with $S = M$), $MF^+ = (F^+)^TM$, or (with $F^+ = A$), $A = MA^TM$. From (27) it follows (with $S = M$, $H = C$) that $CM = C$, and therefore the minimal realization (A,Q,C) satisfies (22).

□

Again we can easily adapt the algorithm of Section 1 if $\Phi(s) = \Phi(-s)$. Indeed there exists a unique $\bar{S}$ satisfying (23). Then we have to find a Σ satisfying (4), (5), (6) *and* $\Sigma = \bar{S}^T\Sigma\bar{S}$. In the basis corresponding to Σ as in (7) we have $\bar{S} = \begin{pmatrix} S & 0 \\ 0 & -S^T\end{pmatrix}$. Finally we have to apply a transformation which changes $K^+ - K^-$ (maximal and minimal solutions of (9)) into I, and S into a signature matrix M.

References

[1] B.D.O. Anderson, T. Kailath, Forwards, backwards and dynamically reversible Markovian models of second order processes, Proc. IEEE Int. Symp. on Circuits and Systems, 1978.

[2] R.W. Brockett, A. Rahimi, Lie algebras and linear differential equations, in: Ordinary Differential Equations (Editor: L. Weiss), Academec Press, New York, 1972, pp. 379-386.

[3] R.E. Kalman, Linear stochastic filtering theory-reappraisal and outlook, Proceedings of the Brooklyn Polytechnic Symposium on System Theory, New York, 1965, pp. 197-205.

[4] A.J. van der Schaft, Time-reversible Hamiltonian systems, Systems & Control Lett. 1, 1982, pp. 295-300.

[5] A.J. van der Schaft, System theoretic descriptions of physical system, Doct. Diss., Groningen, 1983. also: CWI Tracts No. 3, CWI Amsterdam, 1984.

[6] A.J. van der Schaft and J.C. Willems, A new procedure for stochastic realization of spectral density matrices, to appear in SIAM J. Control & Optimization.

[7] M.A. Shayman, A symmetry group for the matrix Riccati equation, Systems & Control Lett. 2, 1982, pp. 17-24.

[8] J.C. Willems, Least squares stationary optimal control and the algebraic Riccati equation, IEEE Trans. Automatic Control, 16, 1971, pp. 621-634.

[9] J.C. Willems, Realization of systems with internal passivity and symmetry constraints, Journal of The Franklin Institute, 301, 1976, pp. 605-621.

[10] J.C. Willems, Time-reversibility in deterministic and stochastic dynamical systems, pp. 318-326, in *Recent Development in Variable Structure Systems, Economics and Biology* (Eds. R.R. Mohler and A.Ruberti), Proc. of US-Italy Seminar, Taormina, Sicily, 1977, Springer Verlag, *Lecture Notes in Economics and Mathematical Systems*, Vol. 162, 1978.